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INVARIANT SOLUTIONS OF THE EQUATION OF NON-STEADY LAMINAR FLOW OF A NON-NEWTONIAN FLUID IN PIPES*

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A method of obtaining particular solutions of the equations of the laminar flow of non-Newtonian fluids in pipes based on a group-theoretic analysis of differential equations is considered. Invariant solutions are given for point and tangential transformations satisfying the natural boundary conditions.

1. The problem of the non-steady laminar flow of a viscous incompressible fluid along a cylindrical pipe can be formulated as the following equation for the rate of flow w:

$$w_t = \nu (w_{rr} + r^{-1}w_r) + \rho^{-1}f(t)$$

where $-\partial p/\partial z = f(t)$ is the given law of variation in the pressure drop and ρ, ν are the density and kinematic viscosity of the fluid, respectively. The above equation has studied in many publications /1-3/ for various dependences of the pressure drop on time.

For a fluid with non-Newtonian properties /4/ the analogous problem is formulated as follows:

$$w_t = \Phi'(w_r) w_{rr} + r^{-1}\Phi(w_r) + \rho^{-1}f(t) \tag{1.1}$$

where Φ is a function characterizing the law of friction of the non-Newtonian fluid.

Making the substitution

$$w = u + \frac{1}{\rho}\int_0^t f(t)\,dt$$

we reduce Eq.(1.1) to the equation

$$u_t = \Phi'(u_r) u_{rr} + r^{-1}\Phi(u_r) \tag{1.2}$$

The aim of the present paper is to obtain certain particular solutions of Eq.(1.2), and hence of (1.1) by studying its group-theoretic properties /5/.

2. In the case of an arbitrary relation $\Phi = \Phi(u_r)$ Eq.(1.2) has a three-dimensional algebra L_3 of infinitesimal operators with the basis $X_1 = \partial/\partial t$, $X_2 = \partial/\partial u$, corresponding to the displacements in t and u, and $X_3 = r\partial/\partial r + 2t\partial/\partial t + u\partial/\partial u$ corresponding to the selfmodelling solution.

The group-theoretic classification of Eq.(1.2) relative to the function Φ, apart from an equivalence transformation /5/, leads to the following result. Extension of the algebra L_3 can occur only for the following specialized $\Phi(u_r)$ (the cases $\Phi' \equiv 0, \Phi' \equiv 1$ are excluded):

1) $\Phi(u_r) = \exp(u_r)$; additional basis operator

$$X_4 = r\partial/\partial r + (u + 2r)\partial/\partial u$$

2) $\Phi(u_r) = u_r^{\lambda}$; additional basis operator

$$X_5 = (\lambda - 1) r\partial/\partial r + (\lambda + 1) u\partial/\partial u$$

3) $\Phi(u_r) = u_r^{-1}$. We obtain an infinitely dimensional group containing, in addition to X_1, X_2, X_3, X_5 (when $\lambda = -1$), the operators

$$X_6 = 8t^2\partial/\partial t + r(u^2 - 2t)\partial/\partial r + 8ut\partial/\partial u$$
$$X_7 = ru\partial/\partial r + 4t\partial/\partial u,\quad X_\infty = \omega r^{-1}\partial/\partial r$$

where the function ω satisfies the equation $\omega_t + \omega_{uu} = 0$.

Let us consider the invariant solutions of Eq.(1.2) connected with the appearance of additional symmetries and corresponding to the natural boundary conditions

$$w|_{r=R} = 0,\quad w_r|_{r=0} = 0 \tag{2.1}$$

where R is the pipe radius (henceforth we shall assume that $R = 1$).

Let us consider the invariant solution for the operator

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$$X_5 + \alpha X_3 = \delta r\partial/\partial r + 2\alpha t\partial/\partial t + \sigma u\partial/\partial u$$
$$\delta = \lambda + \alpha - 1, \quad \sigma = \lambda + \alpha + 1$$

When $\alpha = 0$, we obtain the solution in the form

$$w = \beta(1 - r^{\gamma})(t_0 - t)^{1/(1-\lambda)}, \quad \beta = (\gamma(3\lambda - 1))^{1/(1-\lambda)}, \quad \gamma = (\lambda + 1)/(\lambda - 1)$$

which corresponds to $f(t) = \rho(\lambda - 1)^{-1}(t_0 - t)^{-\lambda/(\lambda-1)}$, satisfies the conditions (2.1) when $\lambda > 1$, and describes a condition with a constraint.

When $\alpha \neq 0$, we can write the invariant solution in the form

$$u = t^{1/2\sigma/\alpha}\varphi(\xi), \quad \xi = rt^{-1/2\sigma/\alpha}$$

where φ satisfies the corresponding differential equation. When $\alpha = -\lambda - 1$, the equation can be integrated in quadratures, and the solution satisfying the conditions (2.1) (when $\lambda < 0$) will be written in the form

$$w = \int\limits_{rt^{-1/(1+\lambda)}}^{t^{-1/(1+\lambda)}} \xi^{-1/\lambda}\left(\frac{1-\lambda}{(1+\lambda)(3\lambda+1)}\xi^{(3\lambda-1)/\lambda} + \xi_0\right)d\xi$$

and will correspond to the condition

$$f(t) = \rho t^{1/[\lambda(\lambda+1)]}\left(\frac{1-\lambda}{(1+\lambda)(3\lambda-1)}t^{(1-3\lambda)/[\lambda(\lambda+1)]}\right)^{1/(\lambda-1)}$$

When $\Phi(u_r) = u_r^{-1}$, we can use the substitution $x = r^2$ to reduce Eq.(1.2) to the form

$$u_t = (1/u_x)_x \tag{2.2}$$

which can be transformed, with help of the substitution $x_1 = u, u_1 = x$, to a linear equation of heat conduction /6, 7/. However, it is not easy to obtain from the solution of this equation, the solutions of Eq.(1.1) satisfying the conditions (2.1) and to study them.

If we introduce the function $v = u_x$, then differentiating (2.2) with respect to x we obtain for this function the equation which has a selfsimilar solution (function φ has a parametric representation):

$$v = {}^1/_2\sqrt{2}\xi\varphi(\ln\xi), \quad \xi = xt^{-1/2} \tag{2.3}$$
$$\varphi = \frac{1}{s - F(s)}, \quad \xi = \exp\int\limits_{s_2}^{s}\frac{(F'(s) - 1)\,ds}{(s - F(s))(sF(s) - F^2(s) - 1)}$$
$$F(s) = \exp\left(\frac{s}{2}\right)^2\left(\int\limits_{s_0}^{s}\exp\left(\frac{\tau^2}{2}\right)d\tau + s_1\right)^{-1}$$

Then, using the inequality (2.3) we can represent the solution of Eq.(1.1) satisfying conditions (2.1), in the form

$$w = \sqrt{t}\int\limits_{\ln r^2t^{-1/2}}^{\ln t^{-1/2}}\frac{dz}{\varphi(z)} \tag{2.4}$$

The solution of Eq.(1.2) corresponding to (2.4) is no longer invariant under point transformation, but will be invariant under some tangential transformation /6/.

3. We will give an example of the construction of some tangential symmetries for Eq.(1.2) discussed in /8/.

Differentiating Eq.(1.2) and introducing a new function $v = u_r$, we obtain the equation

$$v_t = (r^{-1}(r\Phi(v))_r)_r \tag{3.1}$$

The group-theoretic classification of Eq.(3.1) relative to point transformations yields the following result. If $\Phi(v)$ is any function, then Eq.(3.1) will have a two-dimensional algebra with the basis $Y_1 = \partial/\partial t$, $Y_2 = 2t\partial/\partial t + r\partial/\partial r$. The algebra can be extended under the following specializations of $\Phi(v)$ (the case of $\Phi' \equiv 0$, $\Phi' \equiv 1$ is excluded):

1) $\Phi(v) = e^v$; additional operator

$$Y_3 = r\partial/\partial r + 2\partial/\partial v$$

2) $\Phi(v) = v^{\lambda}$; additional operator

$$Y_4 = (\lambda - 1)r\partial/\partial r + 2v\partial/\partial v$$

3) $\Phi(v) = v^{-1/6}$; additional operators

$$Y_4, Y_5 = r^3\partial/\partial r - 5r^2 v\partial/\partial v$$

4) $\Phi(v) = v^{-1}$; additional operators

$$Y_4, Y_6 = r^{-1}\partial/\partial r + vr^{-2}\partial/\partial v$$

The operators $Y_3, \ldots, Y_6$ associate for Eq.(1.2) the operators of tangential symmetry, and make it possible to construct the corresponding invariant solutions.

In particular, when $\Phi(u_r) = u_r^{-1/3}$, the invariant solution corresponding to the operator associated with $Y_2 + {}^5/_6 Y_4 + Y_5$, can be written in the form

$$w = \int_r^1 r \exp\left(\frac{1}{6r^2}\right) F\left(t^{1/2} \exp\left(\frac{1}{6r^2}\right)\right)^{-5} dr$$

where $F(z)$ satisfies the ordinary differential equation

$$2z^2F'' + 6zF' + 15F^{-6}z^{-5}F' + 18F = 0 \tag{3.2}$$

while the following solution corresponds to the operator $Y_2 + Y_5$:

$$w = \int_r^1 (r^2+1)^{-5/2} G^{-5}\left(\frac{r^2}{(r^2+1)t}\right) dr$$

where $G(z)$ satisfies the equation

$$4z^2G'' + 4zG' + 5z^2G^{-6}G' - G = 0 \tag{3.3}$$

Introducing new functions

$$F(z) = z^{-1}\varphi(\ln z), \quad G(z) = z^{1/4}\psi(\ln z)$$

in Eqs.(3.2) and (3.3), we can reduce their order and study them using the methods of the analytic theory of differential equations.

It should be noted that Eq.(1.2) is used in describing the plane-radial filtration of non-Newtonian media /9/. This implies that the results obtained here may also be interpreted from the filtration point of view.

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